An Index for Operational Flexibility in Chemical Process Design

Part I: Formulation and Theory

R. E. SWANEY and
I. E. GROSSMANN
Department of Chemical Engineering
Carnegie-Mellon University
Pittsburgh, PA 15213

One of the key components of chemical plant operability is flexibility—the ability to operate over a range of conditions while satisfying performance specifications. A general framework for analyzing flexibility in chemical process design is presented in this paper. A quantitative index is proposed which measures the size of the parameter space over which feasible steady-state operation of the plant can be attained by proper adjustment of the control variables. The mathematical formulation of this index and a detailed study of its properties are presented. Application of the flexibility in design is illustrated with an example.

SCOPE

The goal in chemical process design is to produce a plant design that is optimal with respect to cost and performance. Plant performance involves a broad range of criteria. A good process design must not only exhibit an optimal balance between capital and operating costs, it must also exhibit operability characteristics which will allow economic performance to be realizable in a practical operating environment. Operability considerations involve flexibility, controllability, reliability, and safety. Although these aspects may appear to be similar, they correspond to different technical concepts. Flexibility is concerned with the problem of insuring feasible steady-state operation over a variety of operating conditions, whereas controllability is concerned with the quality and stability of the dynamic response of the process. Reliability is concerned with the probability of normal operation given that mechanical and electrical failures can occur; safety is concerned with the hazards that are consequences of these failures. Because these operability characteristics are the implicit results of design-stage decisions, they must be given direct attention during the design process if the goal of producing a good design is to be achieved.

Most of the previous methods for process synthesis (Nishida et al., 1981) and flowsheet optimization (Edahl et al., 1983; Biegler and Hughes, 1982; Jirapongphan et al., 1980) consider a single nominal operating condition in the design of chemical processes. Although these procedures can often provide useful results, there is still a substantial gap between the designs obtained from such procedures and the designs that are actually implemented in practice. The major reason for this gap is that conventional procedures for synthesis and flowsheet optimization do not explicitly account for those factors which relate to plant operability. Therefore, the common practice is to introduce additional equipment and employ various types of empirical overdesign to improve operability characteristics. However, with this approach it is generally not possible to guarantee either optimality or feasible operation for conditions that are different from the nominal point selected for the design.

It is only recently that new tools have begun emerging to simultaneously handle both economic and operability aspects in process design (Grossmann et al., 1983; Morari, 1983; Grossmann and Morari, 1984). The purpose of this paper is to present a systematic framework for analyzing flexibility in chemical processes. To accomplish this objective an index of flexibility is proposed which provides a measure of the region of feasible operation in the space of the uncertain parameters. The region considered in this work accounts for the process adjustments that will be made to accommodate different parameter realizations. The proposed index also provides bounds of the parameters within which feasible operation is guaranteed, and it allows the identification of those "worst-case" conditions that limit the flexibility of the process. The mathematical formulation of the proposed index and its basic properties are presented. Application of the index in process design is illustrated with an example. Efficient algorithms for computation of the flexibility index are given in Part II.

CONCLUSIONS AND SIGNIFICANCE

The problem of quantitatively characterizing the flexibility of a chemical plant design has been addressed in this paper. A flexibility index has been proposed which provides a measure of the size of the region of feasible steady-state operation. This index has a meaningful interpretation in that it corresponds to the maximum scaled deviation of uncertain parameters from their nominal values for which feasible operation can be guar-

R. E. Swaney is currently with the Department of Chemical Engineering, University of Wisconsin, Madison, WI.

anteed by proper manipulation of the control variables. Furthermore, computation of this index can be used to identify the critical parameter combinations which limit the feasibility of a given design. A rigorous study of the mathematical properties of the index has been presented, with particular emphasis on the conditions under which it can be guaranteed that the critical points will correspond to extreme values (vertices) of the parameters. Formulations that are useful for the computation of the index have also been presented. Application of the flexibility index has been illustrated in the design of a simple pipe, pump, and control valve system. The example shows that the index can be a useful tool for assessing the flexibility of a design, for comparing alternative flowsheets, and for establishing proper trade-offs between flexibility and cost.

INTRODUCTION

The design procedure may be viewed in two stages: (1) choosing the process configuration, and (2) determining the values for the design parameters of the chosen configuration. In both stages the major objective is to arrive at a design that is both economical and operable. While most systematic design tools that have been developed in the past are directed toward the improvement of process economics, they usually neglect the objective of improving plant operability.

An important first step in incorporating operability considerations at the design stage is to provide an adequate treatment of operational flexibility. Flexibility as a design attribute represents the ability of a design to tolerate and adjust to variations in conditions which may be encountered during operation. The sources of these variations may be both external and internal to the process. Examples of external sources of variations include changes in throughput, feed quality, product requirements, battery-limit conditions, and ambient temperature, as well as fluctuations in utilities. Internal sources include variables such as exchanger fouling and catalyst deactivation. The presence of these variations make it the designer's task to provide a design which will exhibit feasible steady-state operation not just for a particular nominal operating condition, but for a range of varying conditions.

The conventional procedure to provide for flexibility in a design is to choose a conservative set of conditions as the design basis. Additional units are then commonly introduced and empirical overdesign factors are applied to the individual pieces of equipment in the process. While this practice is widely used, it has two major disadvantages. First, since the effects of process interactions on the equipment comprising the process may not be adequately considered, the degree of flexibility actually achieved through this procedure is in general uncertain. Second, since the plant is designed and optimized at a single condition, there is no guarantee that plant performance will be economical over a range of different conditions. Clearly more systematic procedures are desirable.

Recently, Grossmann and Halemane (1982, 1983) have developed procedures for designing optimal chemical plants in which the degree of flexibility is specified either by a finite sequence of discrete operating modes, or by a bounded set of uncertain parameters. This paper addresses the problem of flexibility from a different viewpoint; the objective here is to provide the designer with the capability to:

1. Evaluate the flexibility characteristics of an existing or proposed design in relation to expected operating requirements.
2. Determine the operating conditions which limit the flexibility in a design to identify process bottlenecks.
3. Compare the degrees of flexibility offered by different design *configurations*.

Since a quantitative characterization of flexibility is required to accomplish these objectives, a scalar index for operational flexibility is proposed in this paper. As will be shown, this index provides a measure of the region of feasible operation in the space of the uncertain parameters. An important feature in this work is that this region accounts for the fact that the process can be adjusted depending on the particular parameter realizations. The proposed index also provides parameter bounds for guaranteed feasible operation, as well as information on the "worst" operating conditions that limit the flexibility in a design.

It should be noted that an interesting application of the index of flexibility is that it could be used within a multicriterion optimization framework for minimizing cost and maximizing flexibility as discussed in Grossmann et al. (1983). With this approach trade-off curves for different flowsheets could be obtained as shown in Figure 1 in order to determine optimal degrees of flexibility. This approach circumvents the problem of having to assign economic penalties for infeasible operation, as for instance in the method of Weisman and Holzman (1972). Also, it should be noted that as opposed to the resilience index for heat exchanger networks proposed by Saboo et al. (1985), the index of flexibility presented in this paper is applicable to any chemical processes that operate in the steady state.

Mathematical formulations and basic properties of the index of flexibility are presented here in Part I. The objective is to establish those conditions and properties that can be exploited to simplify the numerical computation of the index. As will be shown in Part II, efficient algorithms can be developed based on the properties presented in this paper.

AN INDEX OF FLEXIBILITY

Flexibility of a plant design represents the ability to accommodate variations of a set of uncertain parameters $\theta_j j = 1, \ldots p$ (e.g., throughput, reaction constants). Since the degree of flexibility is determined by the range of parameter variations that the design can tolerate, a scalar index of flexibility can be constructed to measure the size of the feasible operating region in the space of uncertain parameters θ. The feasible region R shown in Figure 2 gives the complete description of the flexibility characteristics of the design. Combinations of the uncertain parameters lying inside the region permit adjustment of the process to achieve feasible plant operation, while those parameter values outside of the region do not. In general, the actual shape of this region could be rather complex, and since an arbitrary geometry is difficult to treat in a meaningful way, the following approach is proposed.

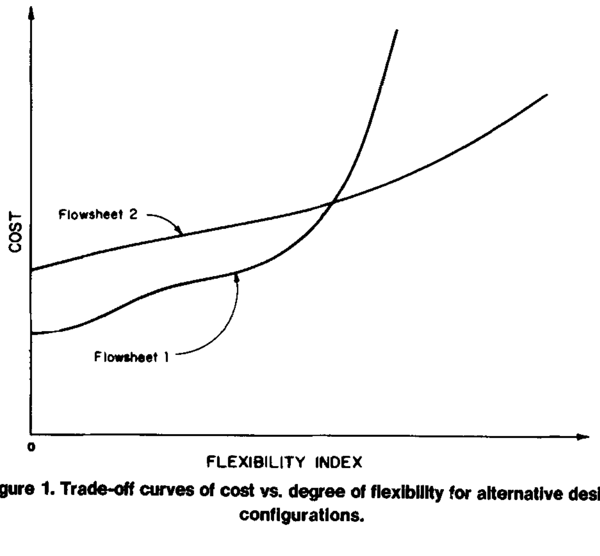

Figure 1. Trade-off curves of cost vs. degree of flexibility for alternative design configurations.

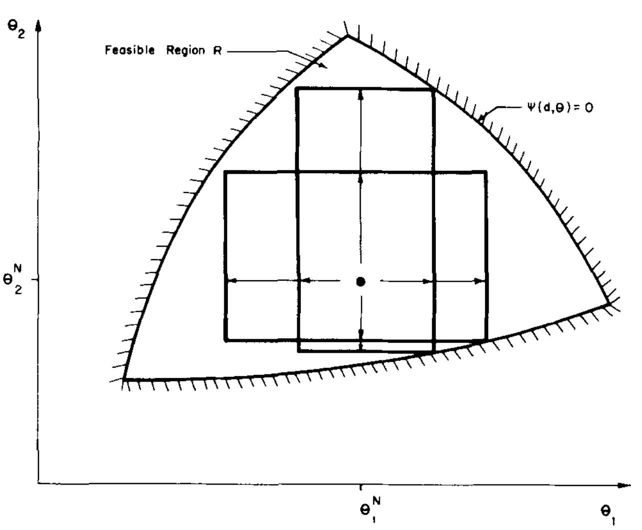

Figure 2. Feasible region in θ-space and inscribed hyperrectangles.

It will be assumed that the uncertain parameters vary independently of each other. (If the set of parameters in the original problem formulation is dependent, then principal component analysis may be employed to obtain an independent set.) It makes sense, then, to analyze the feasible region R by taking feasible nominal parameter values as a base point, and then determining the maximum ranges over which the parameters may vary independently of each other while still remaining inside the feasible region. Geometrically, this approach corresponds to inscribing within the feasible region a hyperrectangle which is centered at the nominal point θ^N as shown in Figure 2. The size of the feasible region is then characterized by the lengths of the sides of the rectangle, which in turn define lower and upper bounds for the parameters. The remaining difficulty is that the rectangle is not uniquely determined; trade-offs can result by increasing the ranges of some parameters while decreasing the ranges of others.

In actual practice, each uncertain parameter θ_j will not vary over totally arbitrary ranges. For example, the temperature of cooling water typically could vary between 290 and 305 K, whereas throughput may vary by as much as ±30% during normal plant operation. Therefore, if it is assumed that expected deviations or range estimates $\Delta\theta_j^+$, $\Delta\theta_j^-$, are given in the positive and negative directions for each parameter j, the sides of the rectangle can be scaled in proportion to the expected deviations. This then yields a unique definition of the maximum hyperrectangle that can be inscribed within the feasible region, as is shown in Figure 3. Using the expected deviations as scaling factors, positive and negative

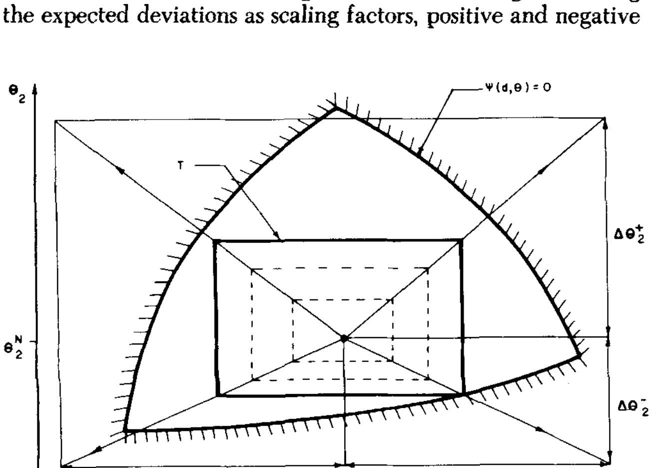

Figure 3. Maximum scaled hyperrectangle T inscribed within feasible region.

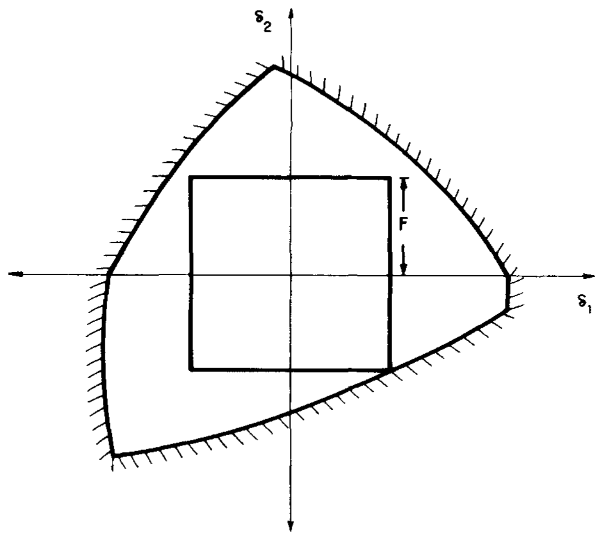

Figure 4. Feasible region and inscribed hypercube in scaled parameter space.

variations in the uncertain parameters may be expressed as scaled deviations from a given nominal value:

$$\delta_j^+ = \frac{\theta_j - \theta_j^N}{\Delta\theta_j^+}, \quad \delta_j^- = \frac{\theta_j^N - \theta_j}{\Delta\theta_j^-}, \quad j = 1, \ldots, p \tag{1}$$

One may then consider the feasible region expressed in the space of the scaled parameters as shown in Figure 4. In the scaled space the hyperrectangle appears as a hypercube, centered at the nominal point (located at the origin). The dimension of the largest hypercube which may be inscribed within the feasible region may then be adopted as the desired measure of the size of the region. The index of flexibility, F, is therefore defined as one-half the length of a side of that hypercube. In that way, any set of scaled parameter deviations which do not exceed the value of F will lie inside the hypercube and therefore permit feasible operation.

Referring back to Figure 3, in the parameter space θ the flexibility index determines the maximum hyperrectangle T that can be expanded around the nominal point with sides proportional to the expected deviations $\Delta\theta_j^+$, $\Delta\theta_j^-$. This hyperrectangle then defines the actual lower and upper bounds, $(\theta^N - F\Delta\theta^-) \le \theta \le (\theta^N + F\Delta\theta^+)$, over which feasible operation can be guaranteed for the set of uncertain parameters.

It should be noted that in the proposed index of flexibility it is through the expected deviations $\Delta\theta_j^+$, $\Delta\theta_j^-$ that the designer specifies that part of the feasible region which is of practical interest. Grossmann et al. (1983) presents a brief discussion of the statistical interpretation of the expected deviations, and suggest that an appropriate choice is to use range or variance estimates of the parameters. In actual practice the design engineer could specify these range estimates based on experience, statistical data, or rule-of-thumb target values.

Mathematical Formulation

Having defined qualitatively the index of flexibility, we now present the mathematical formulation to determine this index. Consider the physical performance of the chemical plant to be described by the following set of constraints

$$\begin{aligned} h(d,z,x,\theta) &= 0 \\ g(d,z,x,\theta) &\le 0 \end{aligned} \tag{2}$$

where h is the vector of equations (such as mass and energy balances or equilibrium relations) which hold for steady-state operation of the process, and g is the vector of inequalities (typically physical operating limits or product specifications) which must be satisfied if operation is to be feasible. The set of all variables is partitioned according to the following scheme: d is the vector of design variables that define equipment sizes. These are fixed at the design stage and remain constant during plant operation. θ is the vector of uncertain parameters referred to previously. The vector of state variables x is a subset of the remaining variables having the same dimension as h. The vector z of control variables represents the degrees of freedom that are available during operation, and which can therefore be adjusted for different realizations of θ.

For a given plant design d, and for any realization of θ during operation, the state variables may be expressed as an implicit function of the control z using the equalities from Eq. 2

$$h(d,z,x,\theta) = 0 \Rightarrow x = x(d,z,\theta)$$

which allows elimination of the state variables, so that the process may be described by the following reduced inequality constraints:

$$g(d,z,x(d,z,\theta),\theta) = f(d,z,\theta) \le 0 \tag{3}$$

The inequalities in Eq. 3 determine feasibility or infeasibility of operation for a chosen control z when d and θ are given. However, since the control variables represent degrees of freedom that may be adjusted during operation to suit prevailing conditions, feasibility for a given d and θ requires only that some z exist for which $f(d,z,\theta) \le 0$. Correspondingly, R, the region of feasibility in θ-space, is defined by

$$R = \{\theta | [\exists z | f(d,z,\theta) \le 0]\} \tag{4}$$

Evaluation of the flexibility index F corresponds to inscribing within region R a scaled hyperrectangle T (see Figure 3) which may be expressed in terms of the nonnegative scalar variable δ as

$$T(\delta) = \{\theta | (\theta^N - \delta\Delta\theta^-) \le \theta \le (\theta^N + \delta\Delta\theta^+)\} \tag{5}$$

The flexibility index, F, for a given design d, is then given by the maximum value of δ in the semiinfinite programming problem

$$F = \max \delta$$
$$\text{s.t. } \forall \theta \in T(\delta) \{\exists z | f(d,z,\theta) \le 0\}$$
$$T(\delta) = \{\theta | (\theta^N - \delta\Delta\theta^-) \le \theta \le (\theta^N + \delta\Delta\theta^+)\} \tag{6}$$

where the first constraint imposes the condition that operation be feasible for all θ values that lie within the hyperrectangle $T(\delta)$. Halemane and Grossmann (1983) have shown that the feasibility condition is mathematically equivalent to

$$\max_{\theta \in T(\delta)} \min_{z} \max_{i \in I} f_i(d,z,\theta) \le 0 \tag{7}$$

Therefore, an equivalent formulation for the flexibility index is

$$F = \max \delta$$
$$\text{s.t. } \max_{\theta \in T(\delta)} \min_{z} \max_{i \in I} f_i(d,z,\theta) \le 0$$
$$T(\delta) = \{\theta | (\theta^N - \delta\Delta\theta^-) \le \theta \le (\theta^N + \delta\Delta\theta^+)\} \tag{8}$$

The direct solution of the problem in Eq. 8 is in general very difficult since it involves the max-min-max constraint, which as discussed in Grossmann et al. (1983) can lead to a nondifferentiable global optimization problem. Before attempting to devise a solution procedure for this problem, it is first necessary to establish some mathematical properties of the flexibility index. In the next three sections, two alternate descriptions of the feasible region R are developed. The first is used to establish conditions under which the solution to Eq. 8 will occur only at a vertex (corner point) of the hyperrectangle T. The second representation provides the foundation for the solution algorithms presented in Part II.

MATHEMATICAL PROPERTIES

Conditions for a Vertex Solution

As shown in the example of Figure 3, the maximum hyperrectangle that can be expanded around the nominal parameter point touches the boundary at a vertex of the hyperrectangle. This vertex can be regarded as a critical point (or worst-case operation) for the design since it is the one that limits the maximum flexibility of the plant (i.e., the size of the maximum hyperrectangle). In the general case there is the possibility of having more than one critical point, since there may be more than one set of conditions which will reach the flexibility limits of a design. Although intuitively one might think that a critical point must lie at a vertex, *this may not always* be the case, as shown in Figure 5, where the maximum hyperrectangle touches the boundary at one of the faces of the hyperrectangle. Therefore, a crucial question arises: What is the nature of the constraint functions for which the critical points will only lie at vertices? For cases where this property holds true, the computation of the index of flexibility could be simplified considerably, as then only the finite number of directions from the nominal point to the vertices would have to be analyzed to determine the maximum hyperrectangle.

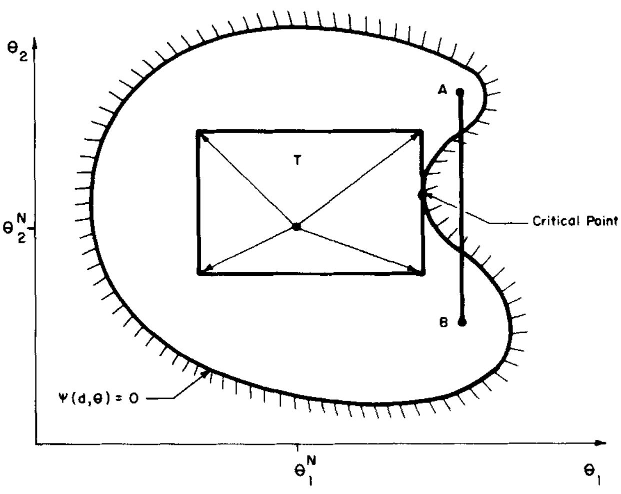

Figure 5. Region with nonvertex critical point.

In order to answer the above question, it is convenient to define the following function which provides a measure of feasibility for a design d at the parameter realization θ (see Halemane and Grossmann, 1983):

$$\psi(d,\theta) = \min_{u,z} u$$
$$\text{s.t. } f_i(d,z,\theta) \le u, i \in I \tag{9}$$

Values of θ for which $\psi(d,\theta) \le 0$ are thus feasible and lie within the region R, which as shown in Appendix I may then also be expressed as

$$R = \{\theta | \psi(d,\theta) \le 0\}. \tag{10}$$

Note that the boundary of R is determined implicitly by the equation $\psi(d,\theta) = 0$ (see Figure 3). Also, the function $\psi(d,\theta)$ can be incorporated into problem 8 by rewriting it as

$$F = \max \delta$$
$$\text{s.t. } \max_{\theta \in T(\delta)} \psi(d,\theta) \le 0$$
$$\psi(d,\theta) = \min_{z} \max_{i \in I} f_i(d,z,\theta)$$
$$T(\delta) = \{\theta | (\theta^N - \delta\Delta\theta^-) \le \theta \le (\theta^N + \delta\Delta\theta^+)\} \tag{11}$$

One first has to determine the properties of the function $\psi(d,\theta)$, which defines the boundary for feasible operation in the parameter space. The following properties, proven in Appendix II, hold for the function $\psi(d,\theta)$:

Property 1. If $f_i(d,z,\theta)$, $i \in I$, are continuous functions in z and θ, then $\psi(d,\theta)$ is a continuous function in θ.

Property 2. If δ^*, θ^* is a bounded solution of problem 11, and $f_i(d,z,\theta)$, $i \in I$, are continuous in z and θ, then $\psi(d,\theta^*) = 0$.

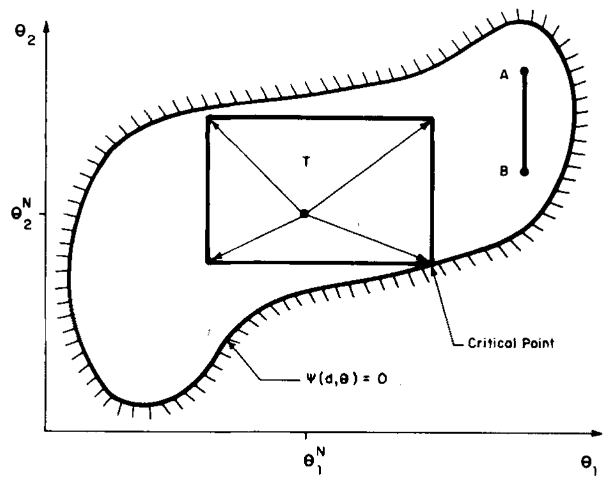

Figure 6. One-dimensional convex region.

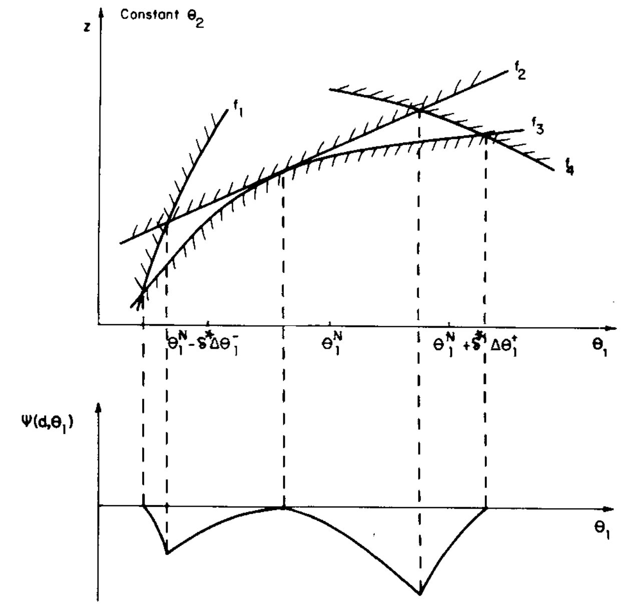

Figure 7. Non-convex subspace for z and parameter θ_1.

Although this second property is rather obvious from Figures 3 and 5, its rigorous proof is important in that it shows that continuity of the constraint functions is a sufficient condition for the solution of Eq. 11 to be at the boundary of region R, $\psi(d,\theta) = 0$.

The next point to be considered concerns the nature of the function $\psi(d,\theta)$ for which the solution of problem 11 will lie at a vertex of the hyperrectangle T. Figure 3 would suggest that a sufficient condition is that $\psi(d,\theta)$ define a convex feasible region. However, this condition is too stringent, since Figure 6 shows a nonconvex region for which the solution still lies at a vertex. On the other hand, the region shown in Figure 5 is also nonconvex, but in this case the solution does not lie at a vertex. As will be shown below, a less stringent condition than requiring convexity of $\psi(d,\theta)$ is that this function be one-dimensional quasi-convex (1-DQC).

Definition 1. $\psi(d,\theta)$ is a 1-DQC function in θ if and only if for $\theta^1,\theta^2 \in R$, where $\theta^2 = \theta^1 + \beta e^j$, β is a nonzero scalar, and e^j is a coordinate direction given by the jth column of the identity matrix I (dim $p \times p$), the following condition holds: $\max\{\psi(d,\theta^1),\psi(d,\theta^2)\} \geq \psi(d,a\theta^1 + (1-a)\theta^2)\ \forall a \in [0,1]$.

Qualitatively what this definition means is that one-dimensional quasi-convexity requires that the function $\psi(d,\theta)$ be quasi-convex only along directions parallel to the coordinates. Therefore, this leaves the possibility that $\psi(d,\theta)$ be not quasi-convex along a direction different from the coordinates. To give a geometrical interpretation of this property it is convenient to define what a one-dimensional convex region is and how it relates to a 1-DQC function.

Definition 2. R is a one-dimensional convex region if and only if for $\theta^1,\theta^2 \in R$, where $\theta^2 = \theta^1 + \beta e^j, \beta \neq 0$, the point $\theta = a\theta^1 + (1-a)\theta^2 \in R$ for all $a \in [0,1]$.

Theorem 1. If $\psi(d,\theta)$ is 1-DQC, then the region R is one-dimensional convex.

In the example shown in Figure 6 one can see that the region R is one-dimensional convex, since by taking any two points belonging to the region and that lie parallel to the coordinates (e.g., points A and B), the line connecting these points also lies inside the region R. However, Figure 5 shows an example for which some points in the line connecting a similar pair of points A and B lie outside the region R. Since in the former case the largest hyperrectangle T touches the boundary at the vertex, it would seem that 1-DQC in the function $\psi(d,\theta)$ is a sufficient condition for this to happen. This property is proved in the following theorem.

Theorem 2. If $\psi(d,\theta)$ is continuous and 1-DQC in θ, then the solution θ^* of problem 11 must lie at an extreme point of the hyperrectangle $T(\delta^*)$.

The significance of this theorem is that it establishes rigorous sufficient conditions under which critical points will correspond to extreme points, or vertices, in the parameter space. However, since this theorem is expressed in terms of requirements for the function $\psi(d,\theta)$, the next relevant question is, what class of constraint functions will lead to a 1-DQC $\psi(d,\theta)$? One could be tempted to think that joint 1-DQC of the parameters θ and controls z in the constraint functions is all that is required. This would be a very desirable property since, for instance, functions that are monotone in z and θ, whether convex or concave, are also 1-DQC. Unfortunately, as shown in the theorem below a somewhat stronger condition must be imposed on the constraint functions to ensure that $\psi(d,\theta)$ is 1-DQC.

Theorem 3. If the constraint functions $f_i(d,z,\theta)$, $i \in I$, are jointly quasi-convex in z and one-dimensional quasi-convex in θ, then the function $\psi(d,\theta)$ is one-dimensional quasi-convex in θ.

The geometrical interpretation of this result is that the constraint functions must define a convex region in the space of the controls and each parameter taken one at a time. If this is the case, then from Theorem 2 it can be guaranteed that the critical point of the hyperrectangle will lie at a vertex. If, on the other hand, a convex region does not exist for one parameter and all the controls, then the critical point may not lie at the vertex of the hyperrectangle. An example of this case is shown in Figure 7, where for constant θ_2 the constraint functions define a nonconvex region in z and θ_1, and the critical point θ^* does not lie at one of the extreme points of the line defined by $\theta_1^N - \delta^*\Delta\theta_1^- \leq \theta_1 \leq \theta_1^N + \delta^*\Delta\theta_1^+$.

Summarizing the results of this section, the solution δ^*,θ^* of problem 8 for the flexibility index is located at the boundary of the feasible region R since $\psi(d,\theta^*) = 0$. Furthermore, if the constraint functions $f_i(d,z,\theta)$ are jointly quasi-convex in z and 1-DQC in θ, then θ^* corresponds to a vertex of the hyperrectangle $T(\delta^*)$. Also, since in this case the maximization of $\psi(d,\theta)$ in the parameter space involves a 1-DQC function, there will be in general more than one local maximum of $\psi(d,\theta)$. Consequently, any algorithm for determining the flexibility index in Eq. 8 which assumes the class of functions described above will in principle have to examine all the vertex directions. It is important to point out that the above results apply also to the case where the constraints $f_i(d,z,\theta)$ are jointly convex in z and θ (which includes, for instance, constraints that are linear in z and θ), since joint convexity is a particular case of one-dimensional quasi-convexity. Also, it should be mentioned that although for practical design problems it might not be possible to establish whether in fact the actual constraints belong to the class of functions described above, the theoretical results presented here describe precisely the class of problems for which the critical points for feasible operation correspond to vertices or extreme values of the parameters.

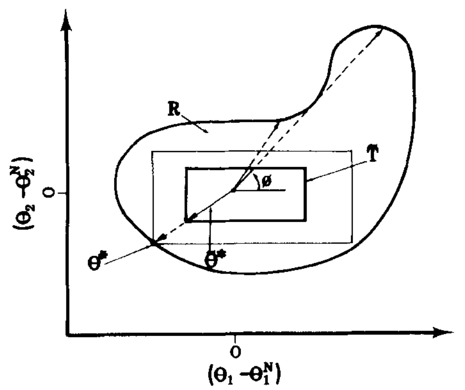

Figure 8a. Parametric description of region R using δ and $\bar{\theta}$.

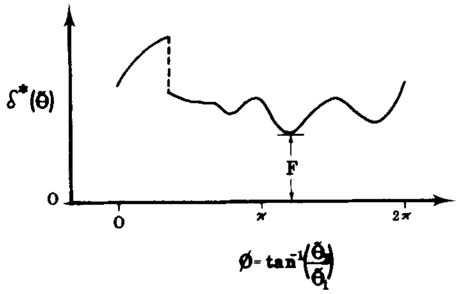

Figure 8b. Corresponding function $\delta^*(\bar{\theta})$.

An Alternate Formulation

In this section a parametric description of the boundary of the feasible region R is developed which leads to an alternate formulation for the flexibility index that is of direct use when developing the solution procedures for problem 8 presented in Part II. The first step is to adopt the following parametric description for θ:

$$\theta = \theta^N + \delta\bar{\theta} \quad (12)$$

The vector $\bar{\theta}$ represents a direction of displacement from the nominal point θ^N, with δ a nonnegative scalar distance (see Figure 8a). By substituting Eq. 12 into Eq. 5, the hyperrectangle T may be expressed in terms of simple bounds for $\bar{\theta}$, i.e., a constant set $\bar{T}$:

$$\bar{T} = \{\bar{\theta} \mid -\Delta\theta^- \leq \bar{\theta} \leq \Delta\theta^+\} \quad (13)$$

The boundary of the feasible region may then be described parametrically as the scaled distance $\delta^*(\bar{\theta})$ from the nominal point to the boundary in the direction $\bar{\theta}$. The function $\delta^*(\bar{\theta})$ is given by

$$\begin{aligned} \delta^*(\bar{\theta}) &= \max_{\delta,z} \delta \\ \text{s.t. } & f(d,z,\theta) \leq 0 \\ & \theta = \theta^N + \delta\bar{\theta} \end{aligned} \quad (14)$$

subject to the following assumption:

Assumption 1. The solution specified by Eq. 14 is taken to be that solution δ^* with the property that

$$\{\exists z \mid f(d,z,(\theta^N + \delta\bar{\theta})) \leq 0\} \,\forall\, \delta \in [0,\delta^*] \quad (15)$$

This assumption provides that the entire path from the nominal point to the boundary at distance δ^* is feasible; i.e., if Eq. 14 has multiple local solutions, the solution with the smallest value for δ is chosen. One may note that since the nominal point ($\delta = 0$) is by definition feasible, Eq. 14 is feasible for all $\bar{\theta}$, and $\delta^*(\bar{\theta}) \geq 0$.

Figures 8a and 8b illustrate the use of Eq. 12 and $\delta^*(\bar{\theta})$ in describing the feasible region. As may be seen, the function $\delta^*(\bar{\theta})$ will not necessarily provide a description of all points on the boundary of region R. Parametric description of the entire boundary would require convexity of R along all radial directions emanating from the nominal point θ^N, a condition which would also guarantee the validity of Assumption 1. However, this convexity restriction is by no means necessary for the purpose at hand, since for any direction $\bar{\theta},\delta^*(\bar{\theta})$ will always describe any point on the boundary of R which could touch the surface of the inscribed hyperrectangle T. Consequently, the utility of the function $\delta^*(\bar{\theta})$ may be seen from the following simple expression for the flexibility index:

Theorem 4. If Assumption 1 holds, then the formulation

$$F = \min_{\bar{\theta} \in \bar{T}} \delta^*(\bar{\theta}) \quad (16)$$

is equivalent to problem 6, where $\delta^*(\bar{\theta})$ is defined in Eq. 14.

Geometrically, solving Eq. 16 consists of finding the direction $\bar{\theta}$ for which the distance $\delta^*(\bar{\theta})$ to the boundary of R is shortest ($\bar{\theta}^*$ in Figure 8a). This then defines the largest hyperrectangle $T(\delta^*)$ that can be inscribed within region R. Note that if the properties of the previous section are met, the solution $\bar{\theta}^*$ will correspond to a vertex direction, i.e., $\bar{\theta}^* \in \{\bar{\theta}^k, k = 1, \ldots, 2^p\}$, where the components $\bar{\theta}_j^k$ are particular choices of either $\Delta\theta_j^+$ or $-\Delta\theta_j^-$.

Properties of $\psi(d,\theta)$ and $\delta^*(\bar{\theta})$

The functions $\psi(d,\theta)$ and $\delta^*(\bar{\theta})$ provide two different means of analyzing the feasible region R, with $\psi(d,\theta)$ representing constraint function values, and $\delta^*(\bar{\theta})$ representing scaled deviations from the nominal point. Below, some properties of these functions are presented and the relation between them is established. The objective is to determine those properties that can be exploited to produce an efficient solution procedure.

First, for convenience $\psi(d,\theta)$ may be reformulated in terms of $\bar{\theta}$ and δ:

$$\begin{aligned} \psi^*(\delta,\bar{\theta}) &= \min_{u,z} u \\ \text{s.t. } & f_i(d,z,\theta) - u \leq 0, \quad i \in I \\ & \theta - \theta^N - \delta\bar{\theta} = 0 \end{aligned} \quad (17)$$

Solving subproblem 17 corresponds to determining feasibility at a particular θ location (i.e., for a particular distance δ in direction $\bar{\theta}$); on the other hand, subproblem 14 defining $\delta^*(\bar{\theta})$ determines the distance δ^* to the boundary of region R along direction $\bar{\theta}$. In the following, the two subproblems are compared for a given direction $\bar{\theta} = \bar{\theta}^k$ (e.g. a vertex direction).

In order to establish the properties of the solutions to Eqs. 17 and 14, it is convenient to formulate the following pair of problems which illustrate the close relation between the two:

P1:
$$\begin{aligned} u^*(\delta) &= \min_{u,z} u \\ \text{s.t. } & f_i(d,z,\theta) - u \leq 0, \quad i \in I \\ & \theta - \theta^N - \delta\bar{\theta}^k = 0 \end{aligned}$$

P2:
$$\begin{aligned} \delta_u^*(u) &= \max_{\delta,z} \delta \\ \text{s.t. } & f_i(d,z,\theta) - u \leq 0, \quad i \in I \\ & \theta - \theta^N - \delta\bar{\theta}^k = 0 \end{aligned}$$

Note from P1 and Eq. 17 that $u^*(\delta) = \psi^*(\delta,\bar{\theta}^k)$, and from P2 and Eq. 14 that $\delta_u^*(0) = \delta^*(\bar{\theta}^k)$. We will make the following two assumptions for problems P1 and P2:

1) Joint quasi-convexity of $f_i(d,z,\theta)$ over the subspace spanned by z and the ray $\theta = \theta^N + \delta\bar{\theta}^k$.

2) Linear independence of the gradients in z of the active constraints.

These assumptions provide that when a solution to P1 or P2 exists, it is determined by the unique solution to the corresponding

Kuhn-Tucker conditions (theorems 4.3.6 and 4.3.7, Bazaraa and Shetty, 1979). The Kuhn-Tucker conditions for P1 may be written:

$$1 - \sum_i \lambda_i = 0 \quad (18a)$$

$$\lambda^T \nabla_z f^T(d,z,\theta) = 0^T \quad (18b)$$

$$\lambda^T \nabla_\theta f^T(d,z,\theta) + v^T = 0^T \quad (18c)$$

$$\lambda \geq 0 \quad (18d)$$

$$\lambda_i[f_i(d,z,\theta) - u] = 0, \quad i \in I \quad (18e)$$

$$f_i(d,z,\theta) - u \leq 0, \quad i \in I \quad (18f)$$

$$\theta - \theta^N - \delta\bar{\theta}^k = 0 \quad (18g)$$

and those for P2:

$$1 + v^T\bar{\theta}^k = 0 \quad (19a)$$

$$\lambda^T \nabla_z f^T(d,z,\theta) = 0^T \quad (19b)$$

$$\lambda^T \nabla_\theta f^T(d,z,\theta) + v^T = 0^T \quad (19c)$$

$$\lambda \geq 0 \quad (19d)$$

$$\lambda_i[f_i(d,z,\theta) - u] = 0, \quad i \in I \quad (19e)$$

$$f_i(d,z,\theta) - u \leq 0, \quad i \in I \quad (19f)$$

$$\theta - \theta^N - \delta\bar{\theta}^k = 0 \quad (19g)$$

Here λ and v are vectors of multipliers for the inequality and equality constraints respectively. Analysis of Eqs. 18 and 19 provides the following conclusions:

Property 3. Let the dimensions of the vectors z and θ be n_z and n_θ respectively. Then n, the number of inequality constraints which are active with nonzero multipliers at the solution to P1 or P2, is given by $n = n_z + 1$.

Property 4. The solutions to both $\delta_u^*(0)$ in P2 and $u^*[\delta_u^*(0)]$ in P1 are characterized by the same set of active constraints and specify identical values for z, θ, u, and δ.

Property 3 implies that a solution to $\psi^*(\delta,\bar{\theta}^k) = u^*(\delta)$ or $\delta^*(\bar{\theta}^k) = \delta_u^*(0)$ is completely determined by solving the system of equations corresponding to the appropriate set of $n_z + 1$ active constraints for z and u or δ. Property 4 shows that the solutions to Eqs. 17 and 14 become identical at the boundary of the feasible region, and that $\psi^*(\delta,\bar{\theta}^k) = 0$. These properties in the form presented depend on linear independence of the z-gradients of the active constraints and on the assumed uniqueness of the solution to the $n_z + 1$ active set equations. Both of those assumptions may be relaxed. In the event that there are more control variables than linearily-independent active constraints, the excess degrees of freedom in z may be specified arbitrarily (e.g., set $z_j = z_j^N$ for those z_j with dependent gradients if their values are not fixed by the stationarity requirements in Eqs. 18b, 19b). When degeneracies occur, uniqueness of the solution may not hold, but the equivalence of solutions to P1 and P2 remains.

The significance of these results is that the subproblem in Eq. 14 can be used to locate the boundary of the region R along a vertex direction to determine the maximum feasible deviation from the nominal point. Then, assuming the solutions to Eq. 8 lie at the vertices of T, problem 16 may be formulated more simply as $F = \min_k\{\delta^*(\bar{\theta}^k)\}$. On the other hand, problem 17 can be used to check for feasibility at a given deviation δ for any vertex k. These subproblems form the basis of the direct search procedure presented in Part II. The example in the next section demonstrates the application of the flexibility index and provides geometric illustration of the mathematical properties established in this paper.

EXAMPLE

Illustration of the flexibility index is provided by the simple example shown in Figure 9. There a centrifugal pump must

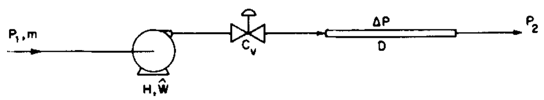

Figure 9. Pump and pipe run example.

transport liquid at a flowrate m from its source at pressure P_1 through a pipe run to its destination at pressure P_2^*. Both the flowrate m and the pressure P_2^* are expected to vary significantly during operation. The actual pressure P_2 must remain within a tolerance ϵ of the desired pressure P_2^*, and is controlled with a valve on the pump discharge. The design variables to be selected are the pipe diameter D, the pump head H, the driver power $\hat{W}$, and the control valve size C_v^{MAX}. The control variable z is the valve coefficient C_v, the uncertain parameters θ_1 and θ_2 are P_2^* and m respectively, and P_2 is a state variable. The nominal value for P_2^* is 800 kPa with expected deviations of +200 and −550 kPa. The nominal value of m is 10 kg/s with expected deviations of +2 and −5 kg/s. P_1 is fixed at 100 kPa. The problem then consists of determining the flexibility index for different designs that could be proposed for this system, as well as developing the trade-off curve of cost versus flexibility.

The equations and constraints describing the above system are as follows:

a) Energy balance

$$P_1 + \rho H - \frac{m^2}{\rho C_v^2} - km^{1.84}D^{-5.16} - P_2 = 0 \quad (20)$$

b) Outlet pressure tolerance

$$P_2^* - \epsilon \leq P_2 \leq P_2^* + \epsilon \quad (21,22)$$

c) Pump driver power limit

$$\frac{mH}{\eta} \leq \hat{W} \quad (23)$$

d) Control valve range

$$rC_v^{MAX} \leq C_v \leq C_v^{MAX} \quad (24,25)$$

where the constant k describes pressure drop in the pipe, the ratio r describes the control valve range, and the liquid density ρ and pump efficiency η are treated as constants. The values of the constants in Eqs. 20–25 are given in Table 1.

The reduced inequalities $f(d,z,\theta) \leq 0$ may be obtained by elimination of the state variable P_2 from Eq. 20, yielding

$$\left(P_1 + \rho H - \frac{m^2}{\rho C_v^2} - km^{1.84}D^{-5.16}\right) - P_2^* - \epsilon \leq 0 \quad (26)$$

$$-\left(P_1 + \rho H - \frac{m^2}{\rho C_v^2} - km^{1.84}D^{-5.16}\right) + P_2^* - \epsilon \leq 0 \quad (27)$$

$$mH - \eta\hat{W} \leq 0 \quad (28)$$

$$C_v - C_v^{MAX} \leq 0 \quad (29)$$

$$rC_v^{MAX} - C_v \leq 0 \quad (30)$$

TABLE 1. DATA FOR EXAMPLE

Pump Efficiency	$\eta = 0.50$
Liquid Density	$\rho = 1000$ kg/m^3
Control Valve Range	$r = 0.05$
Pressure Drop Constant	$k = 9.101 \times 10^{-6}$ (kPa) (kg/s)$^{-1.84}$(m)$^{5.16}$
Inlet Pressure	$P_1 = 100$ kPa
Outlet Pressure Control Tolerance	$\epsilon = 20$ kPa
Annualized Cost	$C = c_1D + c_2\hat{W}^{0.86} + c_3W^N$ $W^N = m^NH/\eta$ $c_1 = 5.6 \times 10^5$ \$/m $c_2 = 482.84$ \$/kW$^{0.86}$ $c_3 = 500$ \$/kW

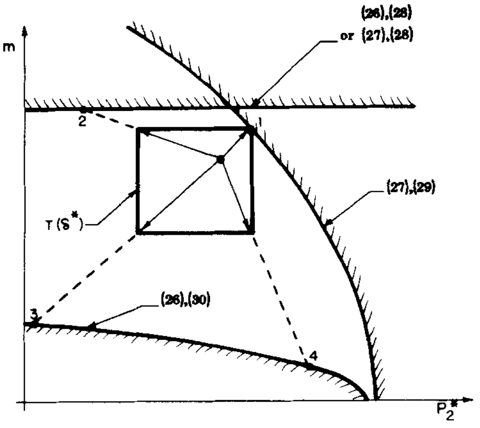

Figure 10. Construction of region R and rectangle $T(\delta^*)$ for pump example.

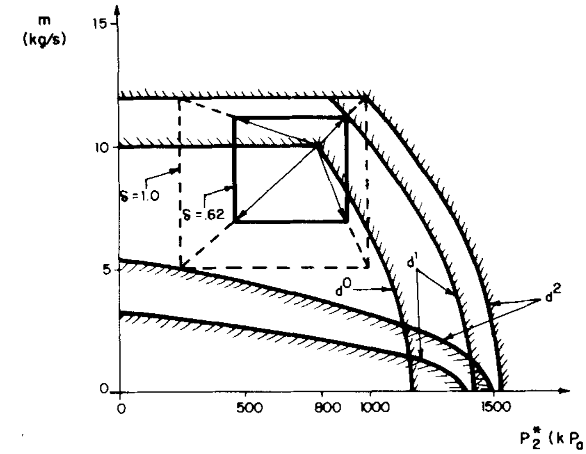

Figure 11. Feasible regions and flexibility indices for designs d^0, d^1, d^2.

For a given design $d = (\hat{W},H,D,C_v^{MAX})$, these inequalities may be rewritten in terms of $z = C_v$, $\theta_1 = P_2^*$, $\theta_2 = m$, and constants a_1–a_6:

$$a_1 - \frac{\theta_2^2}{\rho z^2} - a_2\theta_2^{1.84} - \theta_1 \leq 0 \tag{31}$$

$$-a_3 + \frac{\theta_2^2}{\rho z^2} + a_2\theta_2^{1.84} + \theta_1 \leq 0 \tag{32}$$

$$\theta_2 - a_4 \leq 0 \tag{33}$$

$$z - a_5 \leq 0 \tag{34}$$

$$a_6 - z \leq 0 \tag{35}$$

Since Property 3 applies for this problem, the boundary of the feasible region $\psi(d,\theta) = 0$ may be constructed by considering the appropriate pairs of constraints in Eq. 9 with $u = 0$ and eliminating z between each pair to yield contours in θ-space. Region R is then the intersection of the individual feasible regions for each pair as shown in Figure 10, where the boundary is determined by the constraint pairs Eqs. 27, 29; Eqs. 26, 30; and Eqs. 26, 28 or 27, 28. It is interesting to note that in this example the conditions of Theorem 3 are not met, since the constraint function in Eq. 31 is not jointly quasiconvex in z and each θ_j. However, for this problem the function $\psi(d,\theta)$ is one-dimensional convex in θ in spite of this. Theorems 1 and 2 apply, so that the region R is one-dimensional convex as seen in Figure 10.

Also depicted in Figure 10 is the rectangle $T(\delta^*)$ inscribed within R around the nominal point. As shown, the one-dimensional convexity of region R provides that $T(\delta^*)$, the solution to problem 8, touches the boundary of R at a vertex. Note that Property 3 applies, and therefore $n = n_z + 1 = 2$, since $\delta^*(\bar{\theta}^k)$ at the limiting vertex 1 is determined by the combination of constraints 27 and 29. Vertex 2 illustrates a case of degeneracy. Since $\delta^*(\bar{\theta}^k)$ for vertex 2 is determined by constraint 28, which has a zero z-gradient, linear independence of the gradients is lost. The consequence here is that z is not uniquely determined, and may vary between the limits implied by Eqs. 26 and 27. However, a valid solution to Eq. 14 may still be characterized by a set of $n_z + 1$ active constraints by choosing either Eq. 26 or 27 in conjunction with Eq. 28.

The resulting feasible regions R corresponding to three different proposed designs, as given in Table 2, are depicted in Figure 11. The first design d^0 is a "minimal" design, which has been optimized at the nominal operating point. Since the nominal point lies at the boundary of the feasible region R for d^0, no rectangle of finite size can be expanded, and therefore the flexibility index for this design is zero. Design d^1 is more conventional; pump head ρH has been increased by 230 kPa to reflect the expected 200 kPa increase in P_2^* and to provide an extra 30 kPa for flexibility in the control valve. Driver power $\hat{W}$ is sized at the resulting head and the expected high value for m of 12 kg/s. The index of flexibility for d^1 is illustrated in Figure 11 by the rectangle inscribed within the d^1 region. The critical point for this design lies at the vertex which simultaneously maximizes P_2^* and m, for which $\delta_k = 0.62$. Thus d^1 has a flexibility index of $F = 0.62$, which implies that the maximum variations this design can handle are +1.24 and −3.1 kg/s for the flowrate, and +124 and −341 kPa for the delivery pressure. Therefore, in spite of the chosen overdesign allowances the condition $P_2^* = 1{,}000$ kPa, $m = 12$ kg/s remains infeasible. Nor is d^1 the least expensive design possessing a flexibility index of 0.62, since the driver size could be reduced somewhat without influencing F. Design d^2 shown in Figure 11 is one for which the flexibility targets are exactly met, i.e., $F = 1.0$, at minimum cost as given by the data in Table 1. Note that in this case the region R is modified in such a way so as to exactly contain the rectangle whose sides correspond to the specified expected deviations, +2 and −5 kg/s for the flowrate, and +200 and −550 kPa for the delivery pressure. Also note that in this design three vertices of the rectangle are critical points. This behavior results because the design optimization tries to include the desired rectangle within the smallest possible feasible region, which causes as many vertices to touch the boundary as possible.

As illustrated in Figure 11, the proposed flexibility index provides a systematic measure of the size of that region of feasible steady-state operation which is of interest to the design engineer. Furthermore, this index is very easy to interpret since values that lie between 0 and 1 denote the fraction of the range of expected deviations which can be handled. Values of the index greater than 1 denote designs for which it is possible to exceed the expected deviations and still have feasible operation.

Finally, Figure 12 shows the trade-off curve between cost and flexibility for this system. This curve was generated by optimizing the system for various levels of flexibility (i.e., ϵ-constraint method as in Haimes et al., 1975). The cost function considered is the annualized investment and operating cost which was evaluated at the nominal point condition. As can be seen from this curve, for values of flexibility that lie within 0 and 1.3 only a moderate linear increase is experienced in the cost function. However, for flexibility values greater than 1.3 a rather sharp increase in the cost is experienced since the system becomes less efficient to operate. Qualitatively the reason for this is that as the system is designed to tolerate lower flowrates, the valve introduces a larger pressure drop at the nominal condition. Analysis of this trade-off curve will permit the designer to select an appropriate point in the curve, and so establish the degree of flexibility considered to be optimal.

TABLE 2. DESIGN VARIABLES OF EXAMPLE PROBLEM

	d^0	d^1	d^2
$\hat{W}$, kW	23.75	31.2	33.74
H, kJ/kg	1.187	1.3	1.406
D, m	0.07168	0.0762	0.07643
C_v^{MAX}	5.039	5.77×10^{-3}	2.969×10^{-3}

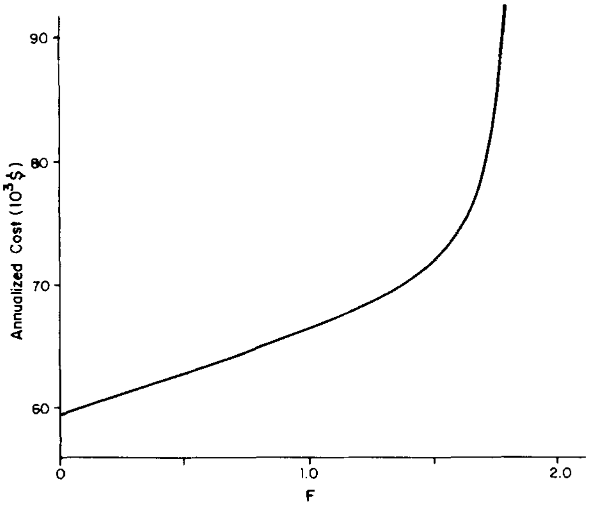

Figure 12. Cost vs. flexibility for pump example.

ACKNOWLEDGMENT

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APPENDIX I

Equivalence for Region Representation

Theorem. The regions $R' = \{\theta | [\exists z | f(d,z,\theta) \le 0]\}$ and $R'' = \{\theta | \psi(d,\theta) \le 0\}$ are equivalent.
Proof. First, by definition

$$R' = \{\theta | [\exists z | f(d,z,\theta) \le 0]\} \equiv \{\theta | [\exists z | \forall i \in I, f_i(d,z,\theta) \le 0]\}$$

Then, by applying global max and min operators, the following equivalences hold:

$$\begin{aligned} R' &= \{\theta | [\exists z | \forall i \in I, f_i(d,z,\theta) \le 0]\} \\ &\Leftrightarrow \left\{\theta \middle| \left[\exists z \middle| \max_{i\in I} f_i(d,z,\theta) \le 0\right]\right\} \\ &\Leftrightarrow \left\{\theta \middle| \min_z \max_{i\in I} f_i(d,z,\theta) \le 0\right\} \\ &\Leftrightarrow \left\{\theta \middle| \left[\min_z u \middle| u \ge f_i(d,z,\theta)\right] \le 0\right\} \\ &\Leftrightarrow \{\theta | \psi(d,\theta) \le 0\} = R'' \end{aligned}$$

which then proves that $R' \equiv R''$.

APPENDIX II

Proofs of Properties and Theorems

Proof of Property 1

The proof of this property can be found in Evans and Gould (1970) and in Polak and Sangiovanni (1979). In Theorem 4.13 of Evans and Gould it is shown that if $\phi(d,z,\theta) = \max_{i\in I} f_i(d,z,\theta)$, then the function $\phi(d,z,\theta)$ is continuous in z and θ. Polak and Sangiovanni show that if $\phi(d,z,\theta)$ is continuous in z and θ then $\psi(d,\theta) = \min_z \phi(d,z,\theta)$ is continuous in θ.

Proof of Property 2

Assume that δ^*,θ^* is a solution of Eq. 11 but that $\psi(d,\theta^*) < 0$. Since for any $\delta' > \delta^*, T(\delta^*) \subset T(\delta')$ it follows that

$$\max_{\theta\in T(\delta^*)} \psi(d,\theta) = \psi(d,\theta^*) \le \max_{\theta \in T(\delta')} \psi(d,\theta) = \psi(d,\theta') \quad \text{(A1)}$$

Since from property 1 $\psi(d,\theta)$ is a continuous function in θ, this implies that there exists $\epsilon > 0$ and a δ'' in the neighborhood of δ^* such that

$$|\psi(d,\theta^*) - \psi(d,\theta'')| < \epsilon$$

where $\psi(d,\theta'') = \max_{\theta\in T(\delta'')} \psi(d,\theta)$. The above inequality and the hypothesis imply

$$\psi(d,\theta'') - \epsilon < \psi(d,\theta^*) < 0$$

which for an arbitrarily small selection of ϵ yields

$$\psi(d,\theta'') < 0 \quad \text{(A2)}$$

But since δ'' can be selected arbitrarily close to δ^*, and such that $\delta'' > \delta^*$, Eqs. A1 and A2 would imply that δ'' is a feasible solution of Eq. 11 for which

$$\psi(d,\theta^*) \le \psi(d,\theta'') < 0$$

which in turn contradicts the assumption that δ^* is the solution to problem 11. Hence $\psi(d,\theta^*) = 0$ at the solution of problem 10.

Proof of Theorem 1

Consider $\theta^1,\theta^2,\theta^2 = \theta^1 + \beta e^j$, $\beta \neq 0$ and such that $\theta^1,\theta^2 \in R$. Assume that $\psi(d,\theta)$ is one-dimensional quasi-convex (1-DQC), but that the region R is not one-dimensional convex. This then implies that there exists an $\bar{a} \in (0,1)$ such that for $\bar{\theta} = \bar{a}\theta^1 + (1-\bar{a})\theta^2$, $\psi(d,\bar{\theta}) > 0$. But since $\psi(d,\theta)$ is 1-DQC we have $\psi(d,\bar{\theta}) \le \max\{\psi(d,\theta^1), \psi(d,\theta^2)\}$.

Furthermore, since $\theta^1,\theta^2 \in R$ implies $\psi(d,\theta^1) \le 0$, $\psi(d,\theta^2) \le 0$, it follows from the above inequality that $\psi(d,\bar{\theta}) \le 0$, which is a contradiction. Therefore, if $\psi(d,\theta)$ is 1-DQC then the region R is one-dimensional convex.

Proof of Theorem 2

Assume that the solution θ^* is a nonextreme point, in which case we can select $\theta^1,\theta^2,\theta^2 = \theta^1 + \beta e^j, \beta \neq 0, \theta^1,\theta^2 \in T(\delta^*)$, and a scalar $\bar{a} \in (0,1)$ such that $\theta^* = \bar{a}\theta^1 + (1-\bar{a})\theta^2$. Since $\psi(d,\theta)$ is 1-DQC this implies that $\psi(d,\theta^*) \le \max\{\psi(d,\theta^1),\psi(d,\theta^2)\}$. But from property 2, $\psi(d,\theta^*) = 0$, which would imply that $\max\{\psi(d,\theta^1),\psi(d,\theta^2)\} \ge 0$.

Clearly, if the equality holds we have a degenerate solution. If the strict inequality holds this would imply that either $\psi(d,\theta^1) > 0$ or $\psi(d,\theta^2) > 0$, which leads to a contradiction. Therefore, θ^* must be an extreme point.

Proof of Theorem 3

(a) In the first part, it will be proved that $\phi(d,z,\theta) = \max_{i\in I} f_i(d,z,\theta)$ is jointly quasi-convex in z and 1-DQC in θ. Assume the negation of this statement, which would then imply that there exists an $\bar{a} \in (0,1)$ such that for z^1,z^2, and $\theta^1,\theta^2,\theta^2 = \theta^1 + \lambda e^j, \lambda \neq 0$, the following inequality applies:

$$\phi(d,\bar{a}y^1 + (1-\bar{a})y^2) > \max\{\phi(d,y^1),\phi(d,y^2)\} \text{ where } y = \begin{bmatrix} z \\ \theta \end{bmatrix}.$$

This in turn implies that there exists a constraint $\bar{i}$, $\phi(d,z,\theta) = f_{\bar{i}}(d,z,\theta)$ such that:

$$f_{\bar{i}}(d,\bar{a}y^1 + (1-\bar{a})y^2) > \max\left\{\max_{i\in I} f_i(d,y^1), \max_{i\in I} f_i(d,y^2)\right\} \ge \max\{f_{\bar{i}}(d,y^1), f_{\bar{i}}(d,y^2)\}.$$

But this contradicts the assumption that the function f_i is jointly quasi-convex in z and 1-DQC in θ. Therefore, $\phi(d,z,\theta)$ is jointly quasi-convex in z, and 1-DQC in θ.

(b) It will be proved that $\psi(d,\theta)$ is 1-DQC. Let $\theta^3 = \bar{a}\theta^1 + (1-\bar{a})\theta^2$ where $\bar{a} \in (0,1)$ and θ^1,θ^2 are defined as above. Also let $\psi(d,\theta^k) = \min_z \phi(d,z,\theta^k) = \phi(d,z^k,\theta^k)$, $k = 1,2,3$. Since $\phi(d,z,\theta)$ is jointly quasi-convex in z, and 1-DQC in θ, and z^3 is the global minimizer of $\phi(d,z,\theta^3)$:

$$\max\{\phi(d,z^1,\theta^1),\phi(d,z^2,\theta^2)\} \ge \phi(d,\bar{a}z^1 + (1-\bar{a})z^2,\theta^3) \ge \phi(d,z^3,\theta^3)$$

or equivalently,

$$\max\{\psi(d,\theta^1),\psi(d,\theta^2)\} \geq \psi(d,\theta^3)$$

which proves that $\psi(d,\theta)$ is 1-DQC in θ.

Proof of Theorem 4

Consider that Eq. 6 may be rewritten as

$$F = \max \delta$$

$$\text{s.t. } T(\delta) \subset R \quad \text{(A3)}$$

Let $\hat{\theta}^*$ be the solution to Eq. 16 with $\delta^*(\hat{\theta}^*) = \hat{\delta}$. For all $\delta > \hat{\delta}$, $T(\delta)$ will contain the point $\theta = \theta^N + \delta\hat{\theta}^*$, which is infeasible for Eq. 14 and thus outside of region R. Consequently, $F \leq \hat{\delta}$. Next, consider that all points contained in $T(\hat{\delta})$ may be expressed as $T(\hat{\delta}) = \{\theta | \theta = \theta^N + \delta\hat{\theta}, \hat{\theta} \in \tilde{T}, \delta \in [0,\hat{\delta}]\}$.

Assumption 1 provides that $\theta = \theta^N + \delta\hat{\theta}$ is feasible for all $\delta \in [0,\delta^*(\hat{\theta})]$, and the minimization in Eq. 16 provides that $\hat{\delta} = \delta^*(\hat{\theta}^*) \leq \delta^*(\hat{\theta})$ for all $\hat{\theta} \in \tilde{T}$. Consequently, $T(\hat{\delta}) \subset R, \hat{\delta}$ is the maximizer for Eq. A3, and $F = \hat{\delta}$ may be determined by solving Eq. 16.

Proof of Property 3

Since the gradients in z are assumed to be linearly independent, this implies from Eq. 18b that the number of nonzero multipliers λ_i in $P1$ is $n > n_z$, since Eq. 18a provides that there exists at least one nonzero multiplier. Similarly, from Eq. 19b the number of nonzero multipliers λ_i in $P2$ is $n > n_z$, since from Eq. 19a $v \neq 0$, which implies in Eq. 19c the existence of at least one nonzero multiplier. Furthermore, since the complete set of active constraints contained in Eqs. 18f, 18g and Eqs. 19f, 19g form a system of $(n + n_\theta)$ equations in $(n_z + 1 + n_\theta)$ variables, for this system to have a solution in general will require $n \leq n_z + 1$. Therefore, $n = n_z + 1$.

Proof of Property 4

Let $(z^*,\theta^*,u^*,\delta^*,\lambda^{(2)},v^{(2)})$ denote the solution to P2, where u^* is a specified value. Also let $c = \Sigma_i \lambda_i^{(2)}$. Since $(z^*,\theta^*,u^*,\delta^*,\lambda^{(2)},v^{(2)})$ is a solution to the system of Eqs. 19a–19g, it can be shown by algebraic substitution that $(z^*,\theta^*,u^*,\delta^*,\lambda^{(1)},v^{(1)})$ will be a solution to Eqs. 18a–18g by letting $\lambda^{(1)} = \lambda^{(2)}/c$ and $v^{(1)} = v^{(2)}/c$. Since solutions to P2 and P1 are unique, their values for z^*, θ^*, u^*, and δ^* are thus identical. Furthermore, since $\lambda^{(1)} = \lambda^{(2)}/c$ and $c > 0$, both solutions are characterized by the same active set. The proof of this property is completed by setting $u^* = 0$ in P2, and $\delta^* = \delta_u^*(0)$ in P1.

NOTATION

C_v	= valve constant, $(\text{kgm}^3/\text{kPa})^5/\text{s}$
d	= vector of design variables
D	= pipe diameter, m
f	= vector of reduced inequalities
F	= flexibility index
g	= vector of inequalities
h	= vector of equations
H	= pump head, kJ/kg
I	= index set for components of vector f
k	= constant for pressure drop
m	= liquid flowrate, kg/s
p	= number of uncertain parameters
P_1,P_2	= inlet and outlet pressures of pipe, kPa
r	= control valve range
R	= region of feasible operation
T	= hyperrectangle for uncertain parameters
$\tilde{T}$	= hyperrectangle for parameter deviations
u	= scalar for maximum constraint value
v	= vector of multipliers
$\hat{W}$	= driver power for pump, kW
x	= vector of state variables
z	= vector of control variables

Greek Letters

δ	= scaled parameter deviation
ϵ	= tolerance for delivery pressure
η	= pump efficiency
θ	= vector of uncertain parameters
θ^N	= nominal value of vector θ
$\hat{\theta}$	= displacement direction from θ^N
$\Delta\theta$	= vector of expected parameter deviations
λ	= vector of multipliers
ρ	= liquid density, kg/m^3
ψ	= feasibility function

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